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Mathematical aspects of Kaluza–Klein gravity

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Abstract

We give a coordinate-free calculation of the Ricci tensor $\overline{\mathrm{Ric}}$ and Ricci scalar $\overline{S}$ for a general Kaluza–Klein metric $\overline{g}$ on a fiber bundle $\pi : E \to M$ over a semi-Riemannian manifold (M, g).

The metric $\overline{g}$ is built from the spacetime metric g, a connection $\sigma : E \times TM \to TE$, and a fiber metric h on the vertical bundle (or internal space) VE of TE. The resulting formulas for $\overline{\mathrm{Ric}}$ and $\overline{S}$ are shown to involve new geometric objects: the gauge Hessian H_h^σ and gauge Laplacian $\Delta^\sigma h$, as well as other globally defined quantities. These formulas appear to be the first global version of the many local coordinate versions existing in the literature. Additionally we isolate a class of fiber metrics h and connections σ for which these formulas reduce considerably in complexity.

The higher-dimensional field equations, $\overline{\mathrm{Ric}} - (1/2)\overline{S}\overline{g} = (1/2)\Lambda\overline{g} + 8\pi\overline{T}$, contain the field equations for gravity and gauge fields, but generally the fields depend on the fiber coordinates. However, this dependence can be eliminated if one restricts attention to principal bundles with equivariant fiber metrics and connections.

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1. Introduction

Recent renewed interest in Kaluza–Klein theory and higher-dimensional gravity has been generated by several new ideas that promise to yield plausible physics when the extra dimensions are considerably larger than the Planck length, and indeed even infinite. One of these ideas originated in the work of [1], which suggests that the largeness of the

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Planck scale, $hcM_{\rm Pl} \sim 10^{16}$ TeV, relative to the electroweak scale arises from the m extra dimensions of the higher-dimensional space $(M \times \mathcal{F}, \overline{g})$ via the calculation:

$$\mathcal{A} = \frac{M_0^{m+2}}{16\pi} \int_{M\times\mathcal{F}} \overline{S}\sqrt{\det \overline{g}}\, \mathrm{d}x^n\, \mathrm{d}y^m = \frac{M_0^{m+2}\, \mathrm{vol}(\mathcal{F})}{16\pi} \int_M S\sqrt{\det g}\, \mathrm{d}x^n.$$

Then from $M_{\rm Pl}^2 = M_0^{m+2}\, \mathrm{vol}(\mathcal{F})$ and $hcM_0 \sim 1$ TeV, one calculates the length size r of the extra dimensions is $r = 10^{-1}$ cm, when $m = 2$, and $r = 10^{-6}$ cm, when $m = 3$ (cf. [26]). The other idea, put forward by Randall and Sundrum (cf. [24,25]), views spacetime as a brane in five-dimensional space $M \times_f \mathcal{F}$, with the warp factor $f(y) = \mathrm{e}^{-2k|y|}$ generating the brane. Then in the above integration over the extra dimensions, one now finds that $\mathrm{vol}(\mathcal{F})$ is replaced by a term with factor $(1 - \mathrm{e}^{-2k\pi r})$, and this mechanism allows for $r = \infty$.

This new activity (also cf. [16]) suggests that a more rigorous examination of the mathematical aspects of Kaluza–Klein theory would be valuable and this paper is directed toward such an analysis in terms of global differential geometry and its natural structures.

Before the renewed interest in the area, most of the work in the period 1970–1990 was influenced by the need to resolve the problems of dimensional reduction and consistency, which to a certain extent are connected with compact extra dimensions, and these concerns may now seem to be of lesser importance. Additionally, it is difficult to find in all of this work, a coordinate-free derivation of the central formulas and higher-dimensional equations. Predominantly, the existing derivations start with some “ansatz” about the local coordinate form of the metric and proceed directly to exhibiting the local coordinate form of the Ricci tensor (which has different forms depending on the author’s choice of coordinates). Doing calculations, in local coordinates, of the Christoffel symbols, the Riemann tensor, and then contracting to get the Ricci tensor can be long and tedious, and while the global (coordinate-free) approach that we use here is no shorter, it does bring out the geometric structures involved and allows one to compare the disparate coordinate versions of the Ricci tensor.

We use a very specific approach for incorporating the geometry of spacetime (M, g) and the gauge field potential into the geometry of the higher-dimensional manifold $(E, \overline{g})$. We take E to be a fiber bundle, $\pi : E \to M$, over M and build the metric $\overline{g}$ from a fiber metric h on the vertical bundle VE and a gauge field potential σ, which we view as a splitting map for the short exact sequence (cf. [2]):

$$VE \hookrightarrow TE \to E \times TM.$$

It may be helpful to put this approach into perspective by considering the different ways in which the geometry of a manifold can be related to the geometry of a higher-dimensional “ambient” manifold.

One way is by *immersion*: $\varepsilon : M \to E$, where ε is an immersion, or better, where ε is an embedding. Then M is considered as a submanifold of E and its Levi–Civita covariant derivative ∇ is related to the Levi–Civita covariant derivative $\overline{\nabla}$ on E by the famous *Gauss* and *Weingarten formulas*:

$$\overline{\nabla}_X Y = \nabla_X Y + \alpha(X, Y), \qquad \overline{\nabla}_X \xi = -A_\xi X + D_X \xi.$$

Here X, Y are tangential vector fields along M and ξ is a normal vector field along M (see [15, p. 15]). Using these to calculate the curvature tensor $\overline{\Omega}$ of $\overline{g}$ leads to the well-known *Gauss and Codazzi equations* relating the tangential component of $\overline{\Omega}$ to the curvature tensor of g and the normal component of $\overline{\Omega}$ to the derivatives of the second fundamental form α [15, pp. 22–25]. These computations are entirely analogous to what we do in Sections 5–7. However, the submanifold setting, $\varepsilon : M \hookrightarrow E$, is not recommended since (1) there is no concept of a covariant derivative with respect to a normal vector field ξ along M, i.e., $\overline{\nabla}_\xi$ is not definable, and (2) there is no clear way of incorporating the gauge fields in the metric $\overline{g}$ on E. For these reasons, the formulation of the higher-dimensional field equations in the submanifold setting seems rather limited.

Another approach is to use a *submersion* $\pi : E \to M$, to relate the geometry of M to that of E. O'Neill [19] had the insight to realize that the situation here is analogous (if not dual) to the case of an immersion—there are analogs of the Gauss, Weingarten, Codazzi formulas and equations. O'Neill's approach and formulas for the curvature tensor of $\overline{g}$ are quite general and for the purposes of that paper were quite useful. But for use in Kaluza–Klein theory one needs more structure, and the special case when E is a fiber bundle has come to be viewed as more beneficial. Hogan [11] applied O'Neill's work to the Kaluza–Klein theory, but uses the general formulas of O'Neill only in the case where the submersion $\pi : E \to M$ is actually a fiber bundle and the metric $\overline{g}$ on E is rather special.

Thus, throughout the paper when we restrict to the fiber bundle case we will be working in a less general setting than the case of a submersion. However, this allows us to take the metric $\overline{g}$ on E to have a more specific form: $\overline{g} = \pi^* g + (1 - \sigma\beta)^* h$, which gives more specific results. Note that O'Neill's results [19, Eqs. {1}–{4}] are more general than ours, but there appears to be no simple way to use his equations to get ours.

The global form of the Riemann tensor $\overline{\Omega}$ leads easily to the global forms of the Ricci tensor $\overline{\mathrm{Ric}}$ and the higher-dimensional field equations (the Kaluza–Klein equations). We show that there is a class of fiber metrics h and gauge field potentials σ for which the Kaluza–Klein equations simplify a great deal. This class (h is *gauge-trivial* with respect to σ) generalizes many standard settings, such as (1) $E = P$, a principal bundle with h the Killing–Cartan metric and σ a principal connection, and (2) product bundles $E = M \times \mathcal{F}$ with any metric h on $\mathcal{F}$ and σ the trivial connection. Lastly, we indicate how including a warp factor on the spacetime part of the metric $\overline{g}$ is necessary to generalize the Randall–Sundrum model to higher-dimensional fibers.

2. Preliminaries

In this section, we establish notation and discuss basic concepts (see [14,15,20] for additional details and concepts not explained here).

We let (M, g) be a semi-Riemannian manifold of dimension n with metric g. In applications, M is spacetime, $n = 4$, and g is a Lorentz metric. In the sequel X, Y, Z denote vector fields on M and, when there is no danger of confusion, we will use a dot for the inner product:

$$X \cdot Y \equiv g(X, Y).$$

For a smooth function $f : M \to \mathbb{R}$, we let $X(f)$ denote the function $X(f)(x) = X_x(f) = X_i(x)(\partial f/\partial x_i)(x)$. We use implied summation on repeated indices.

The *Levi–Civita connection* ∇ (or covariant derivative) determined by a metric g is the unique torsion-free, metric connection on *TM*, i.e.:

$$\nabla_X Y = \nabla_Y X + [X, Y], \tag{1}$$

$$X(Y \cdot Z) = \nabla_X Y \cdot Z + Y \cdot \nabla_X Z \tag{2}$$

for all X, Y, Z. For this paper it is important to note that ∇ is defined (in essence) by Koszul's formula [20, p. 61]:

$$\begin{aligned} 2\nabla_X Y \cdot Z = {} & X(Y \cdot Z) + Y(X \cdot Z) - Z(X \cdot Y) \\ & -X \cdot [Y, Z] - Y \cdot [X, Z] + Z \cdot [X, Y]. \end{aligned} \tag{3}$$

Our convention for the (*Riemann*) *curvature operator* Ω for the connection ∇ is

$$\Omega(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}. \tag{4}$$

The *curvature tensor* is $\Omega(X, Y)Z$, while $R(X, Y, Z, Z') = \Omega(X, Y, Z) \cdot Z'$ is called the *Riemann tensor*. We use Ric and S to denote the Ricci tensor and Ricci scalar, respectively. The *Hessian* for a scalar function f on M is the symmetric bilinear form: $H_f(X, Y) = X(Y(f)) - (\nabla_X Y)(f)$.

We let E denote a fiber bundle over M with standard fiber $\mathcal{F}$ (with $m = \dim(\mathcal{F})$) and projection $\pi : E \to M$ on the base space M. The differential of π is the linear fibered morphism $d\pi : TE \to TM$ whose action on the fibers is: for $K_u \in T_u E$, the tangent vector $d\pi|_u K_u \in T_{\pi(u)} M$ is defined by: $(d\pi|_u K_u)(f) = K_u(f \circ \pi)$.

The *vertical bundle* of E is the subbundle $VE = \{(u, V_u) |\, d\pi|_u V_u = 0\}$ of the tangent bundle *TE*. Thus, a vector field V on E is called a *vertical vector field* if $d\pi|_u V_u = 0$ for every $u \in E$. Equivalently:

$$V(f \circ \pi) = 0$$

for every smooth function f on M. From this, it is easy to show that if V, W are vertical vector fields, then so is $[V, W]$.

3. Connections and gauge fields

We let $E \times TM = \{(u, X_x) | u \in E, X_x \in T_x M, \pi(u) = x\}$ be the fibered product of E and *TM*. This gives us a vector bundle over E, and there is a linear fibered morphism $\beta : TE \to E \times TM$ defined by $\beta(u, K_u) = (u, d\pi|_u K_u)$. It is clear that β is onto and its kernel is *VE*. Thus, we get a short exact sequence:

$$VE \hookrightarrow TE \xrightarrow{\beta} E \times TM$$

of vector bundles over E. Splitting maps which split this short exact sequence on the right are, by definition, potentials for the gauge fields.

Specifically, a *connection* (gauge-field potential) is linear fibered morphism $\sigma : E \times TM \to TE$ such that $\beta\circ\sigma = 1$, where 1 denotes the identity map on $E\times TM$. Schematically:

$$VE \hookrightarrow TE \underset{\overset{\longleftarrow}{\sigma}}{\overset{\overset{\beta}{\longrightarrow}}{}} E\times TM,$$

We use the notation $\sigma((u, X_x)) = (u, \sigma_u(X_x))$, with $\sigma_u(X_x)$ being the tangent vector in T_uE assigned to X_x by σ. Thus, each σ_u is a linear map such that $d\pi|_u\sigma_u(X_x) = X_x$. There are many different approaches to, and abstractions of, the notion of a connection (see [5,14,17,18]), but the one above is natural and convenient for Kaluza–Klein theory.

We will also use the notation $\sigma(X)$ to denote the *horizontal lift* of the vector field X on M to a vector field on E. This lift is defined by $\sigma(X)(u) = \sigma_u(X_{\pi(u)})$. In local coordinates (and implied summation on repeated indices) one has

$$\sigma\left(\frac{\partial}{\partial x_\mu}\right) = \frac{\partial}{\partial \overline{x}_\mu} - A^i_\mu\frac{\partial}{\partial \overline{y}_i},$$

where $(\mathcal{O}, \{x_\mu\}^n_{\mu=1})$, $(\mathcal{U}, \{y_i\}^m_{i=1})$ are charts on $M, \mathcal{F}$, while $\overline{x}_\mu, \overline{y}_i$ are the corresponding coordinate functions on $E|_{\mathcal{O}}$, and $A^i_\mu \equiv -\sigma(\partial/\partial x_\mu)(\overline{y}_i)$. It is important to note that

$$\sigma(X)(f\circ\pi) = X(f)\circ\pi \tag{5}$$

for every smooth function f on M. One immediate consequence of this is $\sigma(X)(\sigma(Y)(f\circ\pi)) = X(Y(f))\circ\pi$, and so

$$[\sigma(X), \sigma(Y)](f\circ\pi) = [X, Y](f)\circ\pi = \sigma([X, Y])(f\circ\pi)$$

for all f. Thus $([\sigma(X), \sigma(Y)]-\sigma([X, Y]))(f\circ\pi) = 0$, for all f. Consequently $[\sigma(X), \sigma(Y)]-\sigma([X, Y])$ is a vertical vector field on E. This leads to the following definition.

Definition 1 (gauge fields). For vector fields X, Y on M, the vector field:

$$F(X, Y) \equiv [\sigma(X), \sigma(Y)] - \sigma([X, Y]) \tag{6}$$

is a *vertical* vector field on E. It is easy to see that F is a skew symmetric, $\mathbb{R}$-bilinear form such that $F(fX, Y) = (f\circ\pi)F(X, Y) = F(X, fY)$, for all smooth functions f on M. This form F is called the *gauge field* associated to the connection σ.

The Lie bracket $[\sigma(X), V]$ of the horizontal lift $\sigma(X)$ and a vertical vector field is particularly significant. Noting that

$$\sigma(X)(V(f\circ\pi)) = \sigma(X)(0) = 0, \qquad V(\sigma(X)(f\circ\pi)) = V(X(f)\circ\pi) = 0,$$

we get that $[\sigma(X), V](f\circ\pi) = 0$. Thus, $[\sigma(X), V]$ is a vertical vector field.

Definition 2 (gauge covariant derivative). For a vector field X on M let ∇^σ_X be the operator:

$$\nabla^\sigma_X V \equiv [\sigma(X), V] \tag{7}$$

which maps vertical vector fields into vertical vector fields. It is easy to see that $(X, V) \mapsto \nabla^\sigma_X V$ is an $\mathbb{R}$-bilinear map, with $\nabla^\sigma_{fX}V = (f\circ\pi)\nabla^\sigma_X V$, and

$$\nabla^\sigma_X(\phi V) = \sigma(X)(\phi)V + \phi\nabla^\sigma_X V$$

for smooth functions f on M and ϕ on E. ∇^σ is the *gauge covariant* derivative associated to the gauge potential σ.

The curvature operator corresponding to ∇^σ is

$$\Omega^\sigma(X, Y) = \nabla^\sigma_X\nabla^\sigma_Y - \nabla^\sigma_Y\nabla^\sigma_X - \nabla^\sigma_{[X,Y]}. \tag{8}$$

A straight-forward calculation using the definition (8) and the Jacobi identity for Lie brackets gives the following natural identity:

$$\Omega^\sigma(X, Y)V = [F(X, Y), V] \tag{9}$$

for vector fields X, Y on M and vertical vector fields V on E.

There is a standard extension of ∇^σ_X, as an operator on vertical vector fields, to an operator on vertical-valued forms. In particular, $\nabla^\sigma_X F$ is defined to be the vertical-valued two-form given by

$$(\nabla^\sigma_X F)(Y, Z) = \nabla^\sigma_X(F(Y, Z)) - F(\nabla_X Y, Z) - F(Y, \nabla_X Z). \tag{10}$$

4. Kaluza–Klein metrics

We consider semi-Riemannian metrics $\overline{g}$ on fiber bundles E over M, which are built from a semi-Riemannian metric g on M, a fiber metric h on the vertical bundle VE and a connection σ. Then $(E, \overline{g})$ is the semi-Riemannian manifold which models higher-dimensional gravity.

Let $S^2(VE) = \{(u, h_u)|u \in E, h_u : V_uE \times V_uE \to \mathbb{R}\}$ be the bundle of symmetric bilinear forms on the vertical bundle $VE \to E$. A (smooth) section $h : E \to S^2(VE)$ of this bundle is called a *symmetric form* on VE.

A *fiber metric h* on VE is a symmetric form which is nondegenerate (i.e., h_u is nondegenerate for every $u \in E$). For vertical vector fields V, W on E, we denote by $h(V, W)$ the smooth function defined by

$$h(V, W)(u) \equiv h_u(V_u, W_u)$$

for every $u \in E$.

Definition 3 (Kaluza–Klein metrics). Suppose g is a metric on M, h is a fiber metric on VE, and $\sigma : E \times M \to TE$ is a connection. Define a metric $\overline{g}$ on E by

$$\overline{g}_u(K_u, K'_u) = g_{\pi(u)}(\mathrm{d}\pi|_u K_u, \mathrm{d}\pi|_u K'_u) + h_u(K_u - \sigma_u\,\mathrm{d}\pi|_u K_u, K'_u - \sigma_u\,\mathrm{d}\pi|_u K'_u), \tag{11}$$

where $K_u, K'_u \in T_uE$.

As shown in [2], there is a certain functoriality to this construction, $(g, h, \sigma) \mapsto \overline{g}^{g,h,\sigma}$, and one can characterize the metrics $\overline{g}$ on E that arise like this.

Conventions and notation. The Kaluza–Klein metric defined above is also expressed as

$$\overline{g} = \pi^* g + (1 - \sigma\beta)^* h, \tag{12}$$

using standard conventions from differential geometry [2,22]. As mentioned above for the metric g, we will also use a dot, $\cdot$, to denote the inner product with respect to the metric $\overline{g}$. So if K, K' are vector fields on E, then $K \cdot K' \equiv \overline{g}(K, K')$. On the other hand $X \cdot Y = g(X, Y)$, for vector fields on M. Also, in some of the ensuing formulas we will use a standard convention of identifying a function f on M with its lift to a function $f \circ \pi$ on E.

It is important to note that from the definition of $\overline{g}$ we get

$$\sigma(X) \cdot \sigma(Y) = (X \cdot Y) \circ \pi, \tag{13}$$

$$\sigma(X) \cdot V = 0, \tag{14}$$

$$V \cdot W = h(V, W) \tag{15}$$

for vector fields X, Y on M and vertical vector fields V, W on E.

At this point, we need to introduce an important new concept, the *gauge Hessian* of h, which will enter prominently in the expressions below for the Ricci tensors. It is analogous to the Hessian of a scalar field but requires a Lie derivative term to make it a symmetric form. For this, note that there is a standard extension of the operator ∇^σ_X (which acts on vertical vector fields) to an operator on symmetric forms h on VE. Specifically: $\nabla^\sigma_X h$ is the symmetric form given by

$$(\nabla^\sigma_X h)(V, W) = \sigma(X)(V \cdot W) - \nabla^\sigma_X V \cdot W - V \cdot \nabla^\sigma_X W. \tag{16}$$

The corresponding curvature operator extends as well:

$$\Omega^\sigma(X, Y)h = \nabla^\sigma_X \nabla^\sigma_Y h - \nabla^\sigma_Y \nabla^\sigma_X h - \nabla^\sigma_{[X,Y]} h. \tag{17}$$

The standard extension of the Lie derivative $\mathcal{L}_U V = [U, V]$ on vertical vector fields to an operator on symmetric forms on VE is

$$(\mathcal{L}_U h)(V, W) = U(V \cdot W) - [U, V] \cdot W - V \cdot [U, W]. \tag{18}$$

A straight-forward computation using the definitions and the identity (10) gives the following result.

Proposition 1. *For vector fields X, Y on M and any fiber metric h on the vertical bundle TE*:

$$\Omega^\sigma(X, Y)h = \mathcal{L}_{F(X,Y)} h. \tag{19}$$

This proposition suggests the following definition.

Definition 4 (gauge Hessian). Suppose $\sigma : E \times TM \to TE$ is a connection and h is a fiber metric on VE. For vector fields X, Y on M, let

$$H^\sigma_h(X, Y) = \nabla^\sigma_X \nabla^\sigma_Y h - \nabla^\sigma_{\nabla_X Y} h - \tfrac{1}{2}\mathcal{L}_{F(X,Y)} h. \tag{20}$$

Then by the above proposition H^σ_h is symmetric: $H^\sigma_h(X, Y) = H^\sigma_h(Y, X)$, and has values in the symmetric forms on VE. We call H^σ_h the *gauge Hessian* of the fiber metric h.

We will also need the following alternative expression for the Lie derivative of h given in (19):

$$\begin{aligned}(\mathcal{L}_U h)(V, W) &= U(V\cdot W) - [U, V]\cdot W - V\cdot [U, W]\\ &= U(V\cdot W) - (\overline{\nabla}_U V - \overline{\nabla}_V U)\cdot W - V\cdot(\overline{\nabla}_U W - \overline{\nabla}_W V)\\ &= \overline{\nabla}_V U\cdot W + \overline{\nabla}_W U\cdot V \end{aligned} \tag{21}$$

for vertical vector fields U, V, W.

5. Decompositions of the covariant derivative

We let $\overline{\nabla}$ denote the Levi–Civita connection (or covariant derivative) associated with a Kaluza–Klein metric $\overline{g}$ (see (12)). It is natural to analyze the actions of the covariant derivative on horizontal lifts $\sigma(X)$ and vertical vector fields V. The results can be decomposed into horizontal and vertical components and this leads to the introduction various new operators between the horizontal and vertical spaces. Specifically as in the following definition.

Definition 5. We use: hor, vert to denote the operators that orthogonally project (relative to the metric $\overline{g}$) vector fields on E onto horizontal, vertical vector fields on E. Suppose X is a vector field on M and V, W are vertical vector fields on E. Let

$$A_V X = -\mathrm{hor}(\overline{\nabla}_V \sigma(X)), \tag{22}$$

$$C_X V = \mathrm{vert}(\overline{\nabla}_V \sigma(X)), \tag{23}$$

$$B(V, W) = -\mathrm{hor}(\overline{\nabla}_V W), \tag{24}$$

$$\nabla'_V W = \mathrm{vert}(\overline{\nabla}_V W). \tag{25}$$

The operator A_V defined in (23) is analogous to the Weingarten map for embedded submanifolds. Also note that operator ∇' defined in (26) is a covariant derivative on the bundle of vertical vector fields and its standard curvature tensor is

$$\Omega'(V, W)U \equiv \nabla'_V\nabla'_W U - \nabla'_W\nabla'_V U - \nabla'_{[V,W]}U.$$

The corresponding "Ricci tensor and Ricci scalar", defined by contracting using a basis of vertical vector fields, are denoted by Ric$'$ and S', respectively. In the case when $E = M\times\mathcal{F}$ is the product of semi-Riemannian manifolds (M, g), $(\mathcal{F}, h)$, then ∇', Ric$'$, S' can be identified with the Levi–Civita connection, Ricci tensor and scalar associated with the metric h.

Theorem 1. *Suppose X, Y are vector fields on M and V, W are vertical vector fields on E. Then*

$$\overline{\nabla}_{\sigma(X)}\sigma(Y) = \sigma(\nabla_X Y) + \tfrac{1}{2}F(X, Y), \tag{26}$$

$$\overline{\nabla}_{\sigma(X)} V = -A_V X + \nabla^{\sigma}_X V + C_X V, \tag{27}$$

$$\overline{\nabla}_V \sigma(X) = -A_V X + C_X V, \tag{28}$$

$$\overline{\nabla}_V W = -B(V, W) + \nabla'_V W. \tag{29}$$

Proof.

(1) To prove formula (27) we first look at $\overline{\nabla}_{\sigma(X)}\sigma(Y) \cdot \sigma(Z)$ and use the identities $\sigma(Y) \cdot \sigma(Z) = (Y \cdot Z) \circ \pi$, $\sigma(X)(f \circ \pi) = X(f) \circ \pi$, and

$$\begin{aligned}\sigma(X) \cdot [\sigma(Y), \sigma(Z)] &= \sigma(X) \cdot (F(Y, Z)) + \sigma([Y, Z]) \\ &= \sigma(X) \cdot \sigma([Y, Z]) = (X \cdot [Y, Z]) \circ \pi.\end{aligned}$$

Now using Koszul's formula (4) for each connection ∇ and $\overline{\nabla}$ gives

$$\begin{aligned}&2\overline{\nabla}_{\sigma(X)}\sigma(Y) \cdot \sigma(Z) \\ &\quad = \sigma(X)(\sigma(Y) \cdot \sigma(Z)) + \sigma(Y)(\sigma(X) \cdot \sigma(Z)) - \sigma(Z)(\sigma(X) \cdot \sigma(Y)) \\ &\qquad -\sigma(X) \cdot [\sigma(Y), \sigma(Z)] - \sigma(Y) \cdot [\sigma(X), \sigma(Z)] + \sigma(Z) \cdot [\sigma(X), \sigma(Y)] \\ &\quad = \{X(Y \cdot Z) + Y(X \cdot Z) - Z(X \cdot Y) - X \cdot [Y, Z] - Y \cdot [X, Z] + Z \cdot [X, Y]\} \circ \pi \\ &\quad = 2(\nabla_X Y \cdot Z) \circ \pi = 2\sigma(\nabla_X Y) \cdot \sigma(Z).\end{aligned}$$

This shows that $\text{hor}(\overline{\nabla}_{\sigma(X)}\sigma(Y)) = \sigma(\nabla_X Y)$. Next consider the vertical component. Using Koszul's formula and the orthogonality of horizontal and vertical vectors, we get

$$\begin{aligned}2\overline{\nabla}_{\sigma(X)}\sigma(Y) \cdot V &= \sigma(X)(\sigma(Y) \cdot V) + \sigma(Y)(\sigma(X) \cdot V) - V(\sigma(X) \cdot \sigma(Y)) \\ &\quad -\sigma(X) \cdot [\sigma(Y), V] - \sigma(Y) \cdot [\sigma(X), V] + V \cdot [\sigma(X), \sigma(Y)] \\ &= V \cdot [\sigma(X), \sigma(Y)] = V \cdot F(X, Y).\end{aligned}$$

This shows that $\text{vert}(\overline{\nabla}_{\sigma(X)}\sigma(Y)) = (1/2)F(X, Y)$.

(2) Formulas (29) and (30) follow from the definitions made above. Formula (28) is derived from formula (29) and the fact that $\overline{\nabla}$ is torsion-free:

$$\overline{\nabla}_{\sigma(X)} V = \overline{\nabla}_V \sigma(X) + [\sigma(X), V] = \overline{\nabla}_V \sigma(X) + \nabla^{\sigma}_X V.$$

This completes the proof. □

6. Calculation of the Riemann tensor

In this section we show how the calculation the Riemann tensor:

$$\overline{R}(K_1, K_2, K_3, K_4) = \overline{\Omega}(K_1, K_2)K_3 \cdot K_4$$

for the Kaluza–Klein metric $\overline{g}$ can be expressed in terms of the Riemann tensors for g and h, together with various natural geometric objects involving g, h and the gauge field F.

However, our ultimate goal is to calculate Ricci tensor $\overline{\text{Ric}}$ for $\overline{g}$ in a coordinate-free way—specifically to calculate

$$\overline{\text{Ric}}(\sigma(X), \sigma(Z)), \qquad \overline{\text{Ric}}(\sigma(X), V), \qquad \text{and} \qquad \overline{\text{Ric}}(V, U),$$

where X, Z are the vector fields on M and U, V, W the vertical vector fields on E. For this, because of the standard identities that allow us to permute the arguments of the Riemann tensor, it suffices to compute the Riemann tensor evaluated at five combinations of horizontal and vertical vector fields: $\overline{\Omega}(\sigma(X), \sigma(Y))\sigma(Z) \cdot \sigma(Y')$, $\overline{\Omega}(\sigma(X), \sigma(Y))\sigma(Z) \cdot V$, $\overline{\Omega}(V, W)U \cdot \sigma(X)$, $\overline{\Omega}(V, W)U \cdot W'$, and $\overline{\Omega}(\sigma(X), V)\sigma(Z) \cdot W$.

A fairly direct calculation gives expressions for these involving the operators A_V, C_X and bilinear forms B, F. Then contracting will easily give $\overline{\mathrm{Ric}}(\sigma(X), \sigma(Z))$, $\overline{\mathrm{Ric}}(\sigma(X), V)$, and $\overline{\mathrm{Ric}}(V, U)$. However, it takes additional work to recast these expressions for $\overline{\mathrm{Ric}}$ so that the first and third are manifestly symmetric in X, Z and V, U, and, more importantly so that all three expressions involve only the gauge fields F, the fiber metric h on the internal space, and various derivatives of F and h. The first step in this is to relate A, B, C to F, h as the following proposition does.

Proposition 2. *Suppose X, Y are vector fields on M and V, W are vertical vector fields on E. Then*

$$A_V X \cdot \sigma(Y) = \tfrac{1}{2} F(X, Y) \cdot V, \tag{30}$$

$$C_X V \cdot W = B(V, W) \cdot \sigma(X) = \tfrac{1}{2}(\nabla^{\sigma}_X h)(V, W). \tag{31}$$

Proof. The first identity follows easily from the definition, the orthogonality of V and $\sigma(Y)$, formula (27), and the fact that $\overline{\nabla}$ is torsion-free and metric:

$$\begin{aligned} A_V X \cdot \sigma(Y) &= -\overline{\nabla}_V \sigma(X) \cdot \sigma(Y) = -(\overline{\nabla}_{\sigma(X)} V + [V, \sigma(X)]) \cdot \sigma(Y) \\ &= -\overline{\nabla}_{\sigma(X)} V \cdot \sigma(Y) = V \cdot \overline{\nabla}_{\sigma(X)} \sigma(Y) = \tfrac{1}{2} F(X, Y) \cdot V. \end{aligned}$$

This calculation also shows why the minus sign is included in the definition of A_V. Next, from the definitions of C and B we get

$$C_X V \cdot W = \overline{\nabla}_V \sigma(X) \cdot W = -\sigma(X) \cdot \overline{\nabla}_V W = \sigma(X) \cdot B(V, W).$$

This is the first part of identity (32). The second part comes from using Koszul's formula and then definition (17):

$$\begin{aligned} 2C_X V \cdot W &= 2\overline{\nabla}_V \sigma(X) \cdot W \\ &= V(\sigma(X) \cdot W) + \sigma(X)(V \cdot W) - W(V \cdot \sigma(X)) \\ &\quad - V \cdot [\sigma(X), W] - \sigma(X) \cdot [V, W] + W \cdot [V, \sigma(X)] \\ &= \sigma(X)(V \cdot W) - V \cdot [\sigma(X), W] + W \cdot [V, \sigma(X)] = (\nabla^{\sigma}_X h)(V, W). \end{aligned}$$

This completes the proof. □

Note: It follows from (32) that C_X is a symmetric operator and B is a symmetric bilinear form:

$$C_X V \cdot W = V \cdot C_X W, \quad B(V, W) = B(W, V). \tag{32}$$

Since B is symmetric we get from Eq. (30) that

$$[V, W] = \overline{\nabla}_V W - \overline{\nabla}_W V = \nabla'_V W - \nabla'_W V \tag{33}$$

for vertical vector fields V, W on E.

Theorem 2 (curvature tensor). *Suppose X, Y, Z are vector fields on M and V, W, U are vertical vector fields on E. Then*

$$\begin{aligned}\overline{\Omega}(\sigma(X), \sigma(Y))\sigma(Z) &= \sigma(\Omega(X, Y)Z) + \tfrac{1}{2}A_{F(X,Z)}Y - \tfrac{1}{2}A_{F(Y,Z)}X + A_{F(X,Y)}Z \\ &\quad + \tfrac{1}{2}(\nabla^\sigma_X F)(Y, Z) - \tfrac{1}{2}(\nabla^\sigma_Y F)(X, Z) + \tfrac{1}{2}C_X F(Y, Z) \\ &\quad - \tfrac{1}{2}C_Y F(X, Z) - C_Z F(X, Y),\end{aligned} \tag{34}$$

$$\begin{aligned}\overline{\Omega}(V, W)U &= \Omega'(V, W)U + \overline{\nabla}_W B(V, U) - \overline{\nabla}_V B(W, U) + B(W, \nabla'_V U) \\ &\quad - B(V, \nabla'_W U) + B([V, W], U).\end{aligned} \tag{35}$$

Note that in (36), the terms $\overline{\nabla}_W B(V, U)$, $\overline{\nabla}_V B(W, U)$ are not resolved into horizontal and vertical components. All the other summands shown in (35) and (36) are either horizontal or vertical.

Proof. Each of these identities follows directly by applying the definition of $\overline{\Omega}$ and using the identities (27)–(30). *Also, in the calculation of* (35) *use the decomposition* $[\sigma(X), \sigma(Y)] = \sigma([X, Y]) + F(X, Y)$ when computing $\overline{\nabla}_{[\sigma(X),\sigma(Y)]}\sigma(Z)$. □

Definition 6. Suppose $\{X_\mu\}_{\mu=1}^n$ is a local basis for the module of vector fields on a chart $(\mathcal{O}, x_\mu, \mu = 1, \dots, n)$ of M. One choice would be $X_\mu = \partial/\partial x_\mu$. As is customary, we let $g_{\mu\nu} = X_\mu \cdot X_\nu$ be the corresponding metric components and let $g^{\mu\nu}$ be the (μ-ν)th entry of the inverse of the matrix $\{g_{\mu\nu}\}_{\mu,\nu=1}^n$. Similarly for a local basis $\{W_i\}_{i=1}^m$ for the module of vertical vector fields on $E|_{\mathcal{O}}$, we define $h_{ij} = W_i \cdot W_j$ and h^{ij}.

There is a standard inner product, which we denote by $\langle\cdot, \cdot\rangle$, on either scalar or vector-valued forms. Here are two examples of this:

$$\langle F, F\rangle = g^{\mu\gamma}g^{\nu\delta}F(X_\mu, X_\nu) \cdot F(X_\gamma, X_\delta), \tag{36}$$

$$\langle \nabla^\sigma_X h, \nabla^\sigma_Y h\rangle = h^{ij}h^{kp}(\nabla^\sigma_X h)(W_i, W_k)(\nabla^\sigma_Y h)(W_j, W_p). \tag{37}$$

These definitions are independent of the local bases $\{X_\mu\}$ and $\{W_i\}$ used.

In the sequel $\nabla^\sigma h$ and $\tilde{F}$ denote the scalar-valued 3-forms defined by $(\nabla^\sigma h)(V, W, X) \equiv (\nabla^\sigma_X h)(V, W)$ and $\tilde{F}(X, Y, V) \equiv F(X, Y) \cdot V$. Similarly, $F \cdot V$ denotes the scalar-valued 2-form: $(F \cdot V)(X, Y) = F(X, Y) \cdot V$.

We use i_X and i_V to denote the contraction operators on tensor fields. For example, $i_X F$ is the 1-form $(i_X F)(Y) = F(X, Y)$ and $i_V \nabla^\sigma h$ is the 2-form: $(i_V \nabla^\sigma h)(W, X) = (\nabla^\sigma h)(V, W, X) = (\nabla^\sigma_X h)(V, W)$.

We can also use local bases to prove the following important identities:

$$\begin{aligned} A_V Z \cdot A_W X &= A_V Z \cdot (g^{\mu\nu}[A_W X \cdot \sigma(X_\nu)]\sigma(X_\mu)) \\ &= \tfrac{1}{4} g^{\mu\nu} F(Z, X_\mu) \cdot V\, F(X, X_\nu) \cdot W = \tfrac{1}{4}\langle i_Z F \cdot V, i_X F \cdot W\rangle. \end{aligned} \tag{38}$$

The first equation above uses the standard basis representations of a horizontal vector field in terms of the basis $\{\sigma(X_\mu)\}_{\mu=1}^{n}$. Similarly, we get

$$\begin{aligned} B(V, U) \cdot B(W, W') &= B(V, U) \cdot (g^{\mu\nu}[B(W, W') \cdot \sigma(X_\nu)]\sigma(X_\mu)) \\ &= \tfrac{1}{4} g^{\mu\nu} (\nabla^\sigma_{X_\mu} h)(V, U)(\nabla^\sigma_{X_\nu} h)(W, W') \\ &= \tfrac{1}{4}\langle i_V i_U \nabla^\sigma h, i_W i_{W'} \nabla^\sigma h\rangle, \end{aligned} \tag{39}$$

$$\begin{aligned} C_Z V \cdot C_X W &= C_Z V \cdot (h^{jk}[C_X W \cdot W_k] W_j) \\ &= \tfrac{1}{4} h^{jk} (\nabla^\sigma_Z h)(V, W_j)(\nabla^\sigma_X h)(W, W_k) = \tfrac{1}{4}\langle i_V \nabla^\sigma_Z h, i_W \nabla^\sigma_X h\rangle. \end{aligned} \tag{40}$$

There are standard exterior derivatives d and exterior co-derivatives ∂ on scalar and vector-valued forms. (For vector-valued forms, a covariant derivative on the vector bundle is needed (see [5, 27, pp. 66–70].) For our purposes here we just need the exterior co-derivatives associated with ∇^σ and ∇'. Specifically, $\partial^\sigma F$ and $\partial' \nabla^\sigma_X h$ are the 1-forms defined by

$$(\partial^\sigma F)(Y) = -g^{\mu\nu} (\nabla^\sigma_{X_\mu} F)(X_\nu, Y), \tag{41}$$

$$(\partial' \nabla^\sigma_X h)(V) = -h^{ij} (\nabla'_{W_i} \nabla^\sigma_X h)(W_j, V). \tag{42}$$

(See (11) and (17) for definitions of $\nabla^\sigma_X F$ and $\nabla^\sigma_X h$.)

For a function f on M, the Laplacian Δf can be defined as the trace of the Hessian of f, i.e., $\Delta f \equiv \mathrm{tr}(H_f)$. For differential forms on M, this relation between the Laplacian and the Hessian is generally not so simple [22,27]. However, for the fiber metric h it turns out to be convenient to define the *gauge Laplacian* of h to be the symmetric form $\Delta^\sigma h$ on *VE* given by

$$\Delta^\sigma h \equiv \mathrm{tr}(H^\sigma_h) = g^{\mu\nu} H^\sigma_h (X_\mu, X_\nu). \tag{43}$$

With all the notation and identities established, we can now prove some of the main results.

Theorem 3 (Riemann tensor part I). *If X, Z are vector fields on M and V, W are vertical vector fields on E, then*

$$\begin{aligned} &\overline{\Omega}(\sigma(X), V)\sigma(Z) \cdot W \\ &= \tfrac{1}{2} H^\sigma_h (X, Z)(V, W) - \tfrac{1}{4}\langle i_W \nabla^\sigma_X h, i_V \nabla^\sigma_Z h\rangle - \tfrac{1}{4}\langle i_X F \cdot W, i_Z F \cdot V\rangle \\ &\quad + \tfrac{1}{4}\nabla'_W F(X, Z) \cdot V - \tfrac{1}{4}\nabla'_V F(X, Z) \cdot W. \end{aligned} \tag{44}$$

Proof. A straight-forward calculation from (27) and (30) gives

$$\overline{\nabla}_{\sigma(X)}\overline{\nabla}_V\sigma(Z) = -\overline{\nabla}_{\sigma(X)}A_VZ - A_{C_ZV}X + \nabla^\sigma_XC_ZV + C_XC_ZV,$$
$$\overline{\nabla}_V\overline{\nabla}_{\sigma(X)}\sigma(Z) = -A_V\nabla_XZ + C_{\nabla_XZ}V - \tfrac{1}{2}B(V, F(X, Z)) + \tfrac{1}{2}\nabla'_VF(X, Z),$$
$$\overline{\nabla}_{[\sigma(X),V]}\sigma(Z) = -A_{[\sigma(X),V]}\sigma(Z) + C_Z\nabla^\sigma_XV.$$

Computing $\overline{\Omega}(\sigma(X), V)\sigma(Z)$ from these and projecting in the vertical direction gives

$$\overline{\Omega}(\sigma(X), V)\sigma(Z)\cdot W = -\overline{\nabla}_{\sigma(X)}A_VZ\cdot W + \nabla^\sigma_XC_ZV\cdot W + C_XC_ZV\cdot W - C_Z\nabla^\sigma_XV\cdot W - C_{\nabla_XZ}\cdot W - \tfrac{1}{2}\nabla'_VF(X, Z)\cdot W. \quad (45)$$

Now since $A_VZ\cdot W = 0$ we can rewrite the first term in (44) as

$$-\overline{\nabla}_{\sigma(X)}A_VZ\cdot W = A_VZ\cdot\overline{\nabla}_{\sigma(X)}W = -A_VZ\cdot A_WX = -\tfrac{1}{4}\langle i_ZF\cdot V, i_XF\cdot W\rangle.$$

The second, third, and fourth terms in (44) can be expressed as follows:

$$\begin{aligned}
&\nabla^\sigma_XC_ZV\cdot W + C_XC_ZV\cdot W - C_Z\nabla^\sigma_XV\cdot W\\
&= (\nabla^\sigma_X + C_X)C_ZV\cdot W - C_Z\nabla^\sigma_XV\cdot W = \overline{\nabla}_{\sigma(X)}C_ZV\cdot W - \tfrac{1}{2}(\nabla^\sigma_Zh)(\nabla^\sigma_XV, W)\\
&= \sigma(X)(C_ZV\cdot W) - C_ZV\cdot\overline{\nabla}_{\sigma(X)}W - \tfrac{1}{2}(\nabla^\sigma_Zh)(\nabla^\sigma_XV, W)\\
&= \tfrac{1}{2}\sigma(X)((\nabla^\sigma_Zh)(V, W)) - C_ZV\cdot(\nabla^\sigma_XW + C_XW) - \tfrac{1}{2}(\nabla^\sigma_Zh)(\nabla^\sigma_XV, W)\\
&= \tfrac{1}{2}\sigma(X)((\nabla^\sigma_Zh)(V, W)) - \tfrac{1}{2}(\nabla^\sigma_Zh)(V, \nabla^\sigma_XW) - \tfrac{1}{2}(\nabla^\sigma_Zh)(\nabla^\sigma_XV, W) - C_ZV\cdot C_XW\\
&= \tfrac{1}{2}(\nabla^\sigma_X\nabla^\sigma_Zh)(V, W) - \tfrac{1}{4}\langle i_V\nabla^\sigma_Zh, i_W\nabla^\sigma_Xh\rangle.
\end{aligned}$$

The fifth term in (44) is

$$-C_{\nabla_XZ}V\cdot W = -\tfrac{1}{2}(\nabla^\sigma_{\nabla_XZ}h)(V, W).$$

Thus the sum of the second through the fifth terms in (44) is

$$\begin{aligned}
&\tfrac{1}{2}(\nabla^\sigma_X\nabla^\sigma_Xh)(V, W) - \tfrac{1}{2}(\nabla^\sigma_{\nabla_XZ}h)(V, W) - \tfrac{1}{4}\langle i_V\nabla^\sigma_Xh, i_W\nabla^\sigma_Zh\rangle\\
&\quad = \tfrac{1}{2}H^\sigma_h(X, Z)(V, W) + \tfrac{1}{4}(\mathcal{L}_{F(X,Z)}h)(V, W) - \tfrac{1}{4}\langle i_V\nabla^\sigma_Zh, i_W\nabla^\sigma_Xh\rangle.
\end{aligned}$$

But $(\mathcal{L}_{F(X,Z)}h)(V, W) = \nabla'_VF(X, Z)\cdot W + \nabla'_WF(X, Z)\cdot V$ (see (22)), and so combining one fourth of this with the last term in (44) gives the last two terms in (45). This accounts for all the terms in (45) and completes the proof. □

Theorem 4 (Riemann tensor part II). *Suppose X, Y, Y', Z are vector fields on M and V, W, W', U are vertical vector fields on E. Then*

$$\overline{\Omega}(\sigma(X), \sigma(Y))\sigma(Z)\cdot\sigma(Y') = \Omega(X, Y)Z\cdot Y' + \tfrac{1}{4}F(Y, Y')\cdot F(X, Z) + \tfrac{1}{4}F(X, Y')\cdot F(Z, Y) + \tfrac{1}{2}F(Z, Y')\cdot F(X, Y), \quad (46)$$

$$\begin{aligned}
\overline{\Omega}(\sigma(X), \sigma(Y))\sigma(Z)\cdot V = {}&\tfrac{1}{2}(\nabla^\sigma_XF)(Y, Z)\cdot V - \tfrac{1}{2}(\nabla^\sigma_YF)(X, Z)\cdot V\\
&+\tfrac{1}{4}(\nabla^\sigma_Xh)(F(Y, Z), V) - \tfrac{1}{4}(\nabla^\sigma_Yh)(F(X, Z), V)\\
&-\tfrac{1}{2}(\nabla^\sigma_Zh)(F(X, Y), V),
\end{aligned} \quad (47)$$

$$\overline{\Omega}(V,W)U\cdot\sigma(X) = \tfrac{1}{4}\langle i_V i_U \nabla^\sigma h, i_X F\cdot W\rangle - \tfrac{1}{4}\langle i_W i_U \nabla^\sigma h, i_X F\cdot V\rangle \tfrac{1}{2}(\nabla'_W \nabla^\sigma_X h)(V,U) - \tfrac{1}{2}(\nabla'_V \nabla^\sigma_X h)(W,U), \tag{48}$$

$$\overline{\Omega}(V,W)U\cdot W' = \Omega'(V,W)U\cdot W' + \tfrac{1}{4}\langle i_V i_U \nabla^\sigma h, i_W i_{W'} \nabla^\sigma h\rangle - \tfrac{1}{4}\langle i_W i_U \nabla^\sigma h, i_V i_{W'} \nabla^\sigma h\rangle. \tag{49}$$

Proof. Eq. (47) follows directly from (35) and the identity (31) applied to terms such as $A_{F(X,Z)}Y\cdot\sigma(Y') = (1/2)F(Y,Y')\cdot F(X,Z)$. Eq. (48) also follows directly from (35) and the identity (32) applied to terms such as $C_X F(Y,Z)\cdot V = (1/2)(\nabla^\sigma_X h)(F(Y,Z),V)$.

To derive (49), we first get directly from (36) that

$$\overline{\Omega}(V,W)U\cdot\sigma(X) = \overline{\nabla}_W B(V,U)\cdot\sigma(X) - \overline{\nabla}_V B(W,U)\cdot\sigma(X) + [B(W,\nabla'_V U) - B(V,\nabla'_W U) + B([V,W],U)]\cdot\sigma(X). \tag{50}$$

But since $\overline{\nabla}$ is a metric connection, we can write the first term in (51) as

$$\begin{aligned}\overline{\nabla}_W B(V,U)\cdot\sigma(X) &= W(B(V,U)\cdot\sigma(X)) - B(V,U)\cdot\overline{\nabla}_W\sigma(X)\\ &= \tfrac{1}{2}W(\nabla^\sigma_X h(V,U)) + B(V,U)\cdot A_W X\\ &= \tfrac{1}{2}W(\nabla^\sigma_X h(V,U)) + \tfrac{1}{4}\langle i_V i_U \nabla^\sigma h, i_X F\cdot W\rangle.\end{aligned}$$

Similarly, $\overline{\nabla}_V B(W,U)\cdot\sigma(X)$ is the same expression with V and W interchanged. Further, we can write

$$\begin{aligned}&[B(W,\nabla'_V U) - B(V,\nabla'_W U) + B([V,W],U)]\cdot\sigma(X)\\ &\quad= \tfrac{1}{2}(\nabla^\sigma_X h)(W,\nabla'_V U) - \tfrac{1}{2}(\nabla^\sigma_X h)(V,\nabla'_W U) + \tfrac{1}{2}(\nabla^\sigma_X h)([V,W],U)\\ &\quad= \tfrac{1}{2}(\nabla^\sigma_X h)(W,\nabla'_V U) + \tfrac{1}{2}(\nabla^\sigma_X h)(\nabla'_V W,V) - \tfrac{1}{2}(\nabla^\sigma_X h)(V,\nabla'_W U) - \tfrac{1}{2}(\nabla^\sigma_X h)(\nabla'_W V,U).\end{aligned}$$

In the last line we used: $[V,W] = \nabla'_V W - \nabla'_W V$ (see (34)). Putting all of these together gives Eq. (49).

To get Eq. (50), take the dot product of (36) with W' to get

$$\begin{aligned}\overline{\Omega}(V,W)U\cdot W' &= \Omega'(V,W)U\cdot W' + \overline{\nabla}_W B(V,U)\cdot W' - \overline{\nabla}_V B(W,U)\cdot W'\\ &= \Omega'(V,W)U\cdot W' - B(V,U)\cdot\overline{\nabla}_W W' + B(W,U)\cdot\overline{\nabla}_V W'\\ &= \Omega'(V,W)U\cdot W' + B(V,U)\cdot B(W,W') - B(W,U)\cdot B(V,W').\end{aligned}$$

Now use identity (40) on this to get Eq. (50). □

7. The Ricci tensor and scalar

We are now able to put all of the previous results together to get the following main theorem.

Theorem 5 (Ricci tensor). *Suppose X, Z are vector fields on M and V, U are vertical vector fields on E. Then the Ricci tensor for the Kaluza–Klein metric* $\overline{g} = \pi^* g + (1 - \sigma\beta)^* h$ *is given by*

$$\overline{\mathrm{Ric}}(\sigma(X), \sigma(Z)) = \mathrm{Ric}(X, Z) + \tfrac{1}{2}\langle i_X F, i_Z F\rangle + \tfrac{1}{2}\mathrm{tr}(H_h^\sigma(X, Z)) - \tfrac{1}{4}\langle \nabla_X^\sigma h, \nabla_Z^\sigma h\rangle, \tag{51}$$

$$\begin{aligned}\overline{\mathrm{Ric}}(\sigma(X), V) &= -\tfrac{1}{2}(\partial^\sigma F)(X)\cdot V + \tfrac{1}{4}\langle \mathrm{tr}\nabla^\sigma h, i_X F\cdot V\rangle + \tfrac{1}{2}\langle i_V\nabla^\sigma h, i_X\tilde{F}\rangle \\ &\quad + \tfrac{1}{2}\mathrm{tr}(\nabla_V'\nabla_X^\sigma h) - \tfrac{1}{2} i_V \partial'\nabla_X^\sigma h,\end{aligned} \tag{52}$$

$$\begin{aligned}\overline{\mathrm{Ric}}(V, U) &= \mathrm{Ric}'(V, U) - \tfrac{1}{4}\langle F\cdot V, F\cdot U\rangle + \tfrac{1}{2}(\Delta^\sigma h)(V, U) \\ &\quad + \tfrac{1}{4}\langle \mathrm{tr}\nabla^\sigma h, i_V i_U\nabla^\sigma h\rangle - \tfrac{1}{2}\langle i_V\nabla^\sigma h, i_U\nabla^\sigma h\rangle.\end{aligned} \tag{53}$$

Note that 'tr' denotes the trace with respect to h, e.g., $\mathrm{tr}(\theta) = h^{ij}\theta(W_i, W_j)$.

Corollary 1 (Ricci scalar). *The Ricci scalar* $\overline{S}$ *for the Kaluza–Klein metric* $\overline{g} = \pi^* g + (1 - \sigma\beta)^* h$ *is given by*

$$\overline{S} = S + S' + \tfrac{1}{4}\langle F, F\rangle + \mathrm{tr}(\Delta^\sigma h) + \tfrac{1}{4}\langle \mathrm{tr}\nabla^\sigma h, \mathrm{tr}\nabla^\sigma h\rangle - \tfrac{3}{4}\langle \nabla^\sigma h, \nabla^\sigma h\rangle, \tag{54}$$

where S, S′ denote the Ricci scalars for g, h, respectively.

These formulas for $\overline{\mathrm{Ric}}$ and $\overline{S}$ generalize and include existing formulas previously developed in the literature. For example, the quite general development and nice discussion by Coquereaux and Jadczyk in [8] gives the local *coordinate* versions of $\overline{\mathrm{Ric}}$ and $\overline{S}$ for the case when $E = P$ is a principal G-bundle with G a compact Lie group (see [8, p. 104]) and for the case when E a fiber bundle with standard fiber $\mathcal{F} = G/H$ a homogeneous space (see [8, p. 161]). Our formulas above exhibit the intrinsic geometric objects and operations involved in their local coordinate expressions.

Additionally, Coquereaux and Jadczyk use Kaluza–Klein metrics $\overline{g}$ which are built from G-invariant pieces (σ and h) and, while this is the preferred method of obtaining dimensional reduction and consistency, our formulas show that no G-invariance is needed at this stage. We offer further comments on this below, but reserve a full discussion for another paper. Our goal here is to delineate the global (coordinate-free) geometric structure of $\overline{\mathrm{Ric}}$ and $\overline{S}$ in a setting that includes the coordinate versions occurring in the literature. Further comparisons to other papers will be given in the following sections.

7.1. The Kaluza–Klein field equations

With the explicit expressions of the Ricci tensor and scalar calculated for $\overline{g} = \pi^* g + (1 - \sigma\beta)^* h$, it is now easy to write out the Kaluza–Klein equations. These equations are just the Einstein equations for $\overline{g}$:

$$\overline{\mathrm{Ric}} - \tfrac{1}{2}\overline{S}\overline{g} = \tfrac{1}{2}\Lambda\overline{g} + 8\pi\overline{T}, \tag{55}$$

where Λ is a cosmological constant. We assume that the stress-energy tensor $\overline{T}$ satisfies: (1) $\overline{T}(\sigma(X), \sigma(Y)) = T(X, Y) \circ \pi$, for some tensor T on M, and (2) $\overline{T}(\sigma(X), V) = 0$, for vector fields X, Y on M and vertical vector field V on E. If we use Eqs. (52)–(55), then it is easy to see that the Kaluza–Klein field equation (56) are equivalent to

$$\begin{aligned}\mathrm{Ric}(X, Z) - \tfrac{1}{2}S(X \cdot Z) &= \tfrac{1}{8}\langle F, F\rangle(X \cdot Z) - \tfrac{1}{2}\langle i_X F, i_Z F\rangle - \tfrac{1}{2}\mathrm{tr}(H_h^\sigma(X, Z)) \\ &\quad + \tfrac{1}{4}\langle \nabla_X^\sigma h, \nabla_Z^\sigma h\rangle + \tfrac{1}{2}(S' + \Lambda + D_h^\sigma)(X \cdot Z) + 8\pi T(X, Z),\end{aligned} \tag{56}$$

$$(\partial^\sigma F)(X) \cdot V = \tfrac{1}{2}\langle \mathrm{tr}\nabla^\sigma h, i_X F \cdot V\rangle + \langle i_V \nabla^\sigma h, i_X \tilde{F}\rangle + \mathrm{tr}(\nabla'_V \nabla_X^\sigma h) - i_V \partial' \nabla_X^\sigma h, \tag{57}$$

$$\begin{aligned}&\mathrm{Ric}'(V, U) - \tfrac{1}{2}S'(V \cdot U) \\ &\quad = \tfrac{1}{4}\langle F \cdot V, F \cdot U\rangle + \tfrac{1}{8}\langle F, F\rangle(V, U) - \tfrac{1}{2}(\Delta^\sigma h)(V, U) - \tfrac{1}{4}\langle \mathrm{tr}\nabla^\sigma h, i_V i_U \nabla^\sigma h\rangle \\ &\qquad + \tfrac{1}{2}\langle i_V \nabla^\sigma h, i_U \nabla^\sigma h\rangle + \tfrac{1}{2}(S + \Lambda + D_h^\sigma)(V \cdot U) + 8\pi T(V, U).\end{aligned} \tag{58}$$

Here we have introduced the notation:

$$D_h^\sigma \equiv \mathrm{tr}(\Delta^\sigma h) + \tfrac{1}{4}\langle \mathrm{tr}\nabla^\sigma h, \mathrm{tr}\nabla^\sigma h\rangle - \tfrac{3}{4}\langle \nabla^\sigma h, \nabla^\sigma h\rangle. \tag{59}$$

Eq. (57) has the form of the ordinary *Einstein* field equation for gravity g with the expression on the right side (minus the cosmological term) forming the stress-energy tensor. This tensor automatically includes terms involving the gauge field F as well as terms involving the fiber metric h. Eq. (58) is the *Yang–Mills* equation for the gauge field. Eq. (59) determines the geometry of the internal space and standard fiber $\mathcal{F}$, and can be considered as the higher-dimensional analog of the *Jordan–Brans–Dicke* scalar equation.

The structure of Eqs. (57)–(59) is exceedingly complex and specific solutions of these equations are hard to come by. Imposing special conditions and restricting to product bundles has been the natural route to deriving solutions. But this can lead to inconsistencies and other problems. For example, there is an interesting class of fiber metrics for which all the quantities $\nabla^\sigma h$, H_h^σ, $\Delta^\sigma h$, $D^\sigma h$ vanish (see the next section). Then Eqs. (57)–(59) reduce to

$$\begin{aligned}\mathrm{Ric}(X, Z) - \tfrac{1}{2}S(X \cdot Z) &= \tfrac{1}{8}\langle F, F\rangle(X \cdot Z) - \tfrac{1}{2}\langle i_X F, i_Z F\rangle \\ &\quad + \tfrac{1}{2}(S' + \Lambda)(X \cdot Z) + 8\pi T(X, Z),\end{aligned} \tag{60}$$

$$\partial^\sigma F = 0, \tag{61}$$

$$\begin{aligned}\mathrm{Ric}'(V, U) - \tfrac{1}{2}S'(V \cdot U) &= \tfrac{1}{4}\langle F \cdot V, F \cdot U\rangle + \tfrac{1}{8}\langle F, F\rangle(V \cdot U) \\ &\quad + \tfrac{1}{2}(S + \Lambda)(V \cdot U) + 8\pi T(V, U).\end{aligned} \tag{62}$$

However, as they stand these equations may have no solution since, for instance, in Eq. (61) the quantities $(1/8)\langle F, F\rangle(X, Z) - (1/2)\langle i_X F, i_Z F\rangle$ and S' will generally depend on the fiber coordinates (i.e., internal space coordinates) while all the other quantities do not. This type of inconsistency was first pointed out in [9] (cf. also [23]), where various avenues for resolving the problem were discussed and these were pursued in numerous ensuing papers by various authors. However, as advocated in [8] (cf. also [7]), perhaps the best remedy is to use invariance under a group action to remove the dependence on the fiber coordinates (cf. [3]). Then the gauge field F and scalar field S' are reduced to fields on the base space M and Eq. (61) presents no problem. Furthermore, this mechanism reduces all the fields in the general Kaluza–Klein equations (57)–(59) to fields on M and in the process these equations retain their same form. This approach also makes Eqs. (57)–(59) physically acceptable since all dependence on the fiber coordinates (the extra dimensions) is eliminated. Thus, there is no need for dimensional reduction.

8. Gauge-trivial fiber metrics

There is a special class of Kaluza–Klein metrics for which the Ricci tensors (52)–(54) and Ricci scalar (55) are particularly simple. This is the class for which the fiber metric h on VE and connection σ satisfy

$$\nabla^{\sigma} h = 0 \quad \text{and} \quad \mathcal{L}_{F(X,Y)} h = 0 \tag{63}$$

for all vector fields X, Y on M. (Cf. definitions (17) and (19).) We call h *gauge-trivial with respect to* σ. As we shall see below, on a principal bundle the Killing–Cartan metric is gauge-trivial with respect to any principal connection. Also on a product manifold $E = M \times \mathcal{F}$, any metric h_0 on $\mathcal{F}$ is gauge-trivial with respect to the trivial connection σ on $M \times \mathcal{F}$ (see further).

For applications, we include a warp factor f in the fiber metric $\tilde{h} = (f^2 \circ \pi)h$ in the next theorem. *Caution*: Here, as throughout the paper, $V \cdot U$, Ric', $\langle i_X F, i_Z F\rangle$, etc., refer to the fiber metric h and not to the scaled metric $\tilde{h}$.

Theorem 6. *For a fiber bundle $\pi : E \to M$ over (M, g), suppose h is a fiber metric on VE which is gauge-trivial with respect to a connection σ (i.e., condition (64) holds). Suppose f is a smooth positive function on M and let*

$$\overline{g} = \pi^* g + (1 - \sigma\beta)^* (f^2 \circ \pi) h \tag{64}$$

be the Kaluza–Klein metric (12) built from g, σ, and the fiber metric $\tilde{h} = (f^2 \circ \pi)h$. Then

$$\overline{\mathrm{Ric}}(\sigma(X), \sigma(Z)) = \mathrm{Ric}(X, Z) + \frac{f^2}{2}\langle i_X F, i_Z F\rangle + \frac{m}{f} H_f(X, Z), \tag{65}$$

$$\overline{\mathrm{Ric}}(\sigma(X), V) = -\frac{f^2}{2}(\partial^{\sigma} F)(X) \cdot V + \frac{(m+2)f}{2} F(X, \nabla f) \cdot V, \tag{66}$$

$$\overline{\mathrm{Ric}}(V, U) = \mathrm{Ric}'(V, U) - \frac{f^4}{4}\langle F \cdot V, F \cdot U\rangle + [f\Delta f + (m-1)|\nabla f|^2](V \cdot U) \tag{67}$$

and

$$\overline{S} = S + \frac{1}{f^2}S' + \frac{f^2}{4}\langle F, F\rangle + \frac{2m}{f}\Delta f + \frac{m(m-1)}{f^2}|\nabla f|^2. \tag{68}$$

Here $m = \dim(\mathcal{F})$ *is the dimension of the standard fiber* $\mathcal{F}$ *of* E *and all the operations on the right sides of the equations are with respect to* g *and* h.

Proof. A straight-forward calculation gives

$$\nabla^\sigma_X \tilde{h} = \frac{2X(f)}{f}\tilde{h} \tag{69}$$

and from this we get

$$H^\sigma_{\tilde{h}}(X, Z) = \left[\frac{2H_f(X, Z)}{f} + \frac{2X(f)Z(f)}{f^2}\right]\tilde{h}. \tag{70}$$

Using these and the fact that $\widetilde{\mathrm{tr}}(\tilde{h}) = m$, it is easy to see that Eq. (52) reduces to Eq. (66). *Note*: For clarity we use $\widetilde{\mathrm{tr}}$ to denote the trace with respect to $\tilde{h}$. Similarly $\tilde{\nabla}'$ is the covariant derivative with respect to $\tilde{h}$.

Further note that since $V((2X(f)/f) \circ \pi) = 0$, we get (suppressing the π in the notation):

$$\begin{aligned}(\tilde{\nabla}'_V \nabla^\sigma_X \tilde{h})(U, W) &\equiv V(\nabla^\sigma_X \tilde{h}(U, W)) - \nabla^\sigma_X \tilde{h}(\tilde{\nabla}'_V U, W) - \nabla^\sigma_X \tilde{h}(U, \tilde{\nabla}'_V W) \\ &= \frac{2X(f)}{f}[V(\tilde{h}(U, W)) - \tilde{h}(\tilde{\nabla}'_V U, W) - \tilde{h}(U, \tilde{\nabla}'_V W)] \\ &= \frac{2X(f)}{f}(\overline{\nabla}_V \overline{g})(U, W) = 0.\end{aligned} \tag{71}$$

(Since $\overline{\nabla}$ is the Levi–Civita covariant derivative for $\overline{g}$.) From this it follows that $\widetilde{\mathrm{tr}}(\tilde{\nabla}'_V \nabla^\sigma_X \tilde{h}) = 0$, and

$$i_V \tilde{\partial}' \nabla^\sigma_X h = \tilde{h}^{ij}(\tilde{\nabla}'_{W_i} \nabla^\sigma_X \tilde{h})(W_j, V) = 0.$$

Next since $\nabla^\sigma \tilde{h} = (2/f)\,\mathrm{d}f \otimes \tilde{h}$, we get $\widetilde{\mathrm{tr}}(\nabla^\sigma \tilde{h}) = (2m/f)\,\mathrm{d}f$ and thus

$$\begin{aligned}\langle \widetilde{\mathrm{tr}} \nabla^\sigma h, i_X F \tilde{\cdot} V\rangle &= \left\langle \frac{2m}{f}\,\mathrm{d}f, f^2 i_X F \cdot V\right\rangle = 2mfg^{\mu\nu}\frac{\partial f}{\partial x_\nu}F\left(X, \frac{\partial}{\partial x_\mu}\right)\cdot V \\ &= 2mf\,F(X, \nabla f)\cdot V.\end{aligned}$$

Similarly one finds $\langle i_V \nabla^\sigma h, i_X \tilde{F}\rangle = fF(X, \nabla f)\cdot V$, and putting these together gives Eq. (53). Calculations in a similar vein give Eqs. (54) and (55). Note: Since $\tilde{h}$ and h differ by the scale factor $f^2 \circ \pi$, which is constant on the fibers of E, one has that $\tilde{\nabla}' = \nabla'$ and so $\widetilde{\mathrm{Ric}}' = \mathrm{Ric}'$. However, $\tilde{S}' = f^{-2}h^{ij}\widetilde{\mathrm{Ric}}'(W_i, W_j) = f^{-2}S'$. □

For gauge-trivial fiber metrics, the Kaluza–Klein field equations (57)–(59) are equivalent to the following equations.

The Kaluza–Klein equations for $\overline{g} = \pi^* g + (1 - \sigma\beta)^*(f^2 \circ \pi)h$, with h gauge-trivial with respect to σ are:

$$
\begin{aligned}
&\mathrm{Ric}(X, Z) - \tfrac{1}{2} S(X \cdot Z) \\
&\quad = \frac{f^2}{8}\langle F, F\rangle(X, Z) - \frac{f^2}{2}\langle i_X F, i_Z F\rangle - \frac{m}{f} H_f(X, Z) \\
&\qquad + \left[\frac{m}{f}\Delta f + \frac{m(m-1)}{2f^2}|\nabla f|^2\right](X \cdot Z) + \frac{1}{2}\left(\frac{1}{f^2}S' + \Lambda\right)(X \cdot Z) + 8\pi T(X, Z),
\end{aligned} \tag{72}
$$

$$
\partial^\sigma F = \frac{(m+2)}{f} i_{\nabla f} F, \tag{73}
$$

$$
\begin{aligned}
&\mathrm{Ric}'(V, U) - \tfrac{1}{2} S'(V \cdot U) \\
&\quad = \frac{f^4}{4}\langle F \cdot V, F \cdot U\rangle + \frac{f^4}{8}\langle F, F\rangle(V \cdot U) \\
&\qquad + \left[(m-1) f \Delta f + \frac{(m-1)(m-2)}{2}|\nabla f|^2\right](V \cdot U) \\
&\qquad + \frac{f^2}{2}(S + \Lambda)(V \cdot U) + 8\pi T(V, U).
\end{aligned} \tag{74}
$$

It is important to note that in the special case when the standard fiber $\mathcal{F}$ of E is one-dimensional, then the above equations hold, but they reduce considerably since $m = 1$, $\mathrm{Ric}' = 0$, and $S' = 0$. Specifically one gets the following equations.

The Kaluza–Klein equations for a line bundle (with h gauge-trivial with respect to σ):

$$
\begin{aligned}
\mathrm{Ric}(X, Z) - \tfrac{1}{2} S(X \cdot Z) = {} & \frac{f^2}{8}\langle F, F\rangle(X \cdot Z) - \frac{f^2}{2}\langle i_X F, i_Z F\rangle + \frac{1}{f}\Delta f\,(X \cdot Z) \\
& - \frac{1}{f} H_f(X, Z) + \tfrac{1}{2}\Lambda(X \cdot Z) + 8\pi T(X, Z),
\end{aligned} \tag{75}
$$

$$
\partial^\sigma F = \frac{3}{f} i_{\nabla f} F, \tag{76}
$$

$$
0 = \frac{3f^4}{8}\langle F, F\rangle + \frac{f^2}{2}(S + \Lambda) + 8\pi T'. \tag{77}
$$

Here $T'(V, U) \equiv T(V, U)/(V \cdot U)$. Note that since

$$
\overline{S} = S + \frac{f^2}{4}\langle F, F\rangle + \frac{2}{f}\Delta f,
$$

the last equation (78) can be rewritten as

$$
8\pi T' = f\Delta f - \frac{f^2}{4}\langle F, F\rangle - \frac{f^2}{2}(\overline{S} + \Lambda). \tag{78}
$$

As mentioned in the previous section, the line bundle equations ($m = 1$) as well as the more general equations (73)–(75) (for $m \geq 1$), will generally be inconsistent since F and S' in (76) (and in (73)) can depend on the fiber coordinates. However, this problem is resolved and each set of equations is perfectly valid if $E = P$ is a principal G bundle, h is equivariant, and σ is a principal connection (cf. Sections 9.1 and [3]). In the line bundle case, the geometry of the internal space is more or less predetermined, and we have written Eq. (79) to reflect that T' should be determined from solutions of Eqs. (76) and (77). Similarly, in the nonline bundle case, if we restrict to the case when the Lie group G is semisimple and use the Killing–Cartan metric for h, then the internal space geometry is predetermined and the left-side of (75) reduces

$$\mathrm{Ric}'(V, U) - \tfrac{1}{2} S'(V, U) = -\frac{m-2}{8}(V \cdot U).$$

So in this case too, we view Eq. (75) as determining the stress-energy tensor $T(U, V)$ for the internal space. This is a common strategy when building cosmological models that satisfy the Einstein equations, however for the Kaluza–Klein equations it is unclear at this point what stress-energy means in a virtual internal space.

In the case when the higher-dimensional model is vacuum gravity ($\overline{T} = 0$), with no cosmological constant ($\Lambda = 0$), then $\overline{S} = 0$ as well, and Eqs. (76), (77) and (79) are identical with those in [21, Eq. (6)]. Our derivation shows that it is not necessary to assume the “cylinder condition” on the fields in order to get these equations. More precisely, the fiber metric can depend on the fiber coordinates $\{y_i\}_{i=1}^{m}$, and derivatives with respect to these coordinates are not set to zero. These derivatives occur in Eqs. (73)–(75) only in the terms involving F, since it is calculated from $\sigma(\partial/\partial x_\mu) = (\partial/\partial x_\mu) - A^i_\mu(x, y)(\partial/\partial y_i)$. In the more general Kaluza–Klein equations (57)–(59), derivatives with respect to the fiber coordinates occur as well in all the terms involving h. It is the gauge-triviality that causes these terms to vanish in (73)–(75). Also the spacetime part $\pi^* g$ of the Kaluza–Klein metric is, by construction, independent of the fiber coordinates.

We should also mention in the line bundle case and a vacuum model ($\overline{T} = 0$) with $\Lambda = 0$, Eq. (79) reduces to

$$\Delta f = \frac{f^3}{4}\langle F, F\rangle.$$

The initial inconsistency problem arose from this equation when Kaluza felt compelled to eliminate the unwanted scalar field f by setting $f = 1$. But this yields the unwanted constraint $\langle F, F\rangle = 0$ on the gauge fields.

9. Principal bundles and warped products

There are two special cases for the fiber bundle E which simplify the formulas and thus occur often in the literature. The first is that of a principal bundle $E = P$ with Lie group G for the standard fiber, and this is the natural choice since then the gauge field F is one of the customary ones from Yang–Mills theory. Further one can use equivariance with respect to G to eliminate the consistency problems and achieve dimensional reduction. The second case is that of a product, or warped product, bundle $E = M \times_f \mathcal{F}$. Such bundles admit trivial connections σ—ones with vanishing gauge fields.

9.1. Principal bundles

Suppose $\pi : P \to M$ is a principal G bundle over M, and let $\mathcal{G}$ be the Lie algebra of left-invariant vector fields on G. For $\xi \in \mathcal{G}$, the adjoint action of ξ is the linear map $Ad_\xi : \mathcal{G} \to \mathcal{G}$, given by Lie brackets: $Ad_\xi(\eta) \equiv [\xi, \eta]$. The *negative* of the *Killing–Cartan* form is the symmetric bilinear form h_0 on $\mathcal{G}$ defined by

$$h_0(\xi, \xi') = -\mathrm{tr}(Ad_\xi Ad_{\xi'}),$$

where tr is the canonical trace function for linear operators on a finite-dimensional vector space. We assume that G is semisimple so that, by definition, h_0 is nondegenerate and gives rise to a fiber metric h on the vertical bundle *VP* defined as follows. For $u \in P$, let $\lambda_u : G \to P$ be the map: $\lambda_u(a) \equiv ua$. The differential of this map at the identity $e \in G$ is an injection: $\mathrm{d}\lambda_u|_e : T_eG \to T_uP$, with image V_uP. Since $\mathcal{G} \cong T_eG$, we can define h by

$$h_u(K_u, K'_u) \equiv h_0(\mathrm{d}\lambda_u|_e^{-1} K_u, \mathrm{d}\lambda_u|_e^{-1} K'_u), \tag{79}$$

where $K_u, K'_u \in V_uP$ are vertical vector fields. We call h the *Killing–Cartan fiber metric*.

In the setting of principal bundles, a connection $\sigma : P \times TM \to TP$ is called a *principal connection* if σ is a equivariant map, i.e., if $\sigma(wa) = \sigma(w)a$, for every $a \in G$. To explain this, denote the right action of G on P by $R_a u \equiv ua$, for $u \in P$, $a \in G$. Then, the right action of G on $P \times TM$ is $(u, X_x)a \equiv (ua, X_x)$, and the right action of G on TP is $(u, K_u)a \equiv (ua, \mathrm{d}R_a|_u K_u)$. Thus, equivariance of σ is seen to be equivalent to the property

$$\mathrm{d}R_a|_u \sigma_u = \sigma_{ua} \tag{80}$$

for all $a \in G$ and $u \in P$. Yet a third way to say this is that $\sigma(X)$ is invariant under push-forwards by right multiplications:

$$(R_a)_* \sigma(X) = \sigma(X) \tag{81}$$

for all $a \in G$ and vector fields X on M. This follows from the definition of the push-forward: $[(R_a)_*\sigma(X)](u) = \mathrm{d}R_a|_{ua^{-1}} \sigma_{ua^{-1}}(X)$.

As the next proposition shows, equivariance guarantees that the Killing–Cartan fiber metric is trivial with respect to σ.

Proposition 3. *If σ is a principal connection and h the Killing–Cartan fiber metric on VP, then h is gauge-trivial with respect to σ.*

Proof. We first recall some basic Lie group facts (cf. [14, p. 15, 51, 78]). For each vector field $\xi \in \mathcal{G}$ in the Lie algebra of G there is a vertical vector field $\bar{\xi}$ on P defined by $\bar{\xi}_u \equiv \mathrm{d}\lambda_u|_e \xi_e$. ($\bar{\xi}$ is the fundamental vector associated to ξ.) Further, if ψ_t is the flow generated by ξ, and if we let $a(t) = \psi_t(e)$, where e is the identity of G, then $R_{a(t)}$ is the flow generated by $\bar{\xi}$. Thus, the Lie bracket of $\bar{\xi}$ with any vector field K on P is given by

$$[\bar{\xi}, K] = -\frac{\mathrm{d}}{\mathrm{d}t}[(R_{a(t)})_* K]|_{t=0},$$

where $*$ denotes push-forward under a diffeomorphism. In particular, by equivariance of σ, Eq. (82) gives

$$\nabla^{\sigma}_{X}\bar{\xi} \equiv [\bar{\xi}, \sigma(X)] = -\frac{\mathrm{d}}{\mathrm{d}t}[(R_{a(t)})_{*}\sigma(X)]|_{t=0} = -\frac{\mathrm{d}}{\mathrm{d}t}[(\sigma(X))]|_{t=0} = 0 \tag{82}$$

for each fundamental vector field $\bar{\xi}$ on P and vector field X on M. From this we get

$$(\nabla^{\sigma}_{X}h)(\bar{\xi}, \bar{\eta}) = \sigma(X)(\bar{\xi} \cdot \bar{\eta}) - \nabla^{\sigma}_{X}\bar{\xi} \cdot \bar{\eta} - \bar{\xi}\nabla^{\sigma}_{X}\bar{\eta} = 0. \tag{83}$$

The first term in the above equation is zero since for fundamental vector fields

$$\bar{\xi} \cdot \bar{\eta} = h(\bar{\xi}, \bar{\eta}) = h_0(\xi, \eta)$$

is a constant (i.e., independent of $u \in P$). Furthermore, using the Jacobi identity for Lie brackets gives

$$\begin{aligned}[F(X, Y), \bar{\xi}] &= [[\sigma(X), \sigma(Y)], \bar{\xi}] - [\sigma([X, Y]), \bar{\xi}] = [[\sigma(X), \sigma(Y)], \bar{\xi}] \\ &= -[[\sigma(Y), \bar{\xi}], \sigma(X)] - [[\bar{\xi}, \sigma(X)], \sigma(Y)] = 0.\end{aligned}$$

Consequently,

$$(\mathcal{L}_{F(X,Y)}h)(\bar{\xi}, \bar{\eta}) = F(X, Y)(\bar{\xi}, \bar{\eta}) - [F(X, Y), \bar{\xi}] \cdot \bar{\eta} - \bar{\xi} \cdot [F(X, Y), \bar{\eta}] = 0. \tag{84}$$

Eqs. (84) and (85) show that $\nabla^{\sigma}_{X}h$ and $\mathcal{L}_{F(X,Y)}h$ are zero when evaluated on fundamental vector fields. But since $V_uP = \{\bar{\xi}_u | \xi \in \mathcal{G}\}$, it follows that $\nabla^{\sigma}_{X}h = 0$ and $\mathcal{L}_{F(X,Y)}h = 0$. □

Thus, the results of the previous section, in particular the Kaluza–Klein equations (73)–(75), hold for $\bar{g} = \pi^{*}g + (1 - \sigma\beta)^{*}(f^2 \circ \pi)h$, with h the Killing–Cartan fiber metric. For the sake of comparison with other results in the literature, we record in the next corollary the special case of this when there is no warp factor ($f = 1$).

Corollary 2 (Killing–Cartan fiber metric). *Suppose P is a principal bundle over M with standard fiber G a semisimple Lie group. Let h be the Killing–Cartan fiber metric on VP and $\sigma : P \times TM \to TP$ a principal connection. Then the Ricci tensor and scalar for the Kaluza–Klein metric*:

$$\bar{g} = \pi^{*}g + (1 - \sigma\beta)^{*}h$$

are given by

$$\overline{\mathrm{Ric}}(\sigma(X), \sigma(Z)) = \mathrm{Ric}(X, Z) + \tfrac{1}{2}\langle i_X F, i_Z F\rangle, \tag{85}$$

$$\overline{\mathrm{Ric}}(\sigma(X), V) = \tfrac{1}{2}(\partial^{\sigma} F)(X) \cdot V, \tag{86}$$

$$\overline{\mathrm{Ric}}(V, U) = \mathrm{Ric}'(V, U) - \tfrac{1}{4}\langle F \cdot V, F \cdot U\rangle \tag{87}$$

and

$$\bar{S} = S + S' + \tfrac{1}{4}\langle F, F\rangle. \tag{88}$$

The formulas (86)–(89), when written in local coordinates, give the formulas found in many papers in the literature. For example, the second set of Cho's formulas [6, p. 2034, formulas (23)–(24)] (also cf. [13]) easily result by using the local bases $\{\sigma(\partial/\partial x_\mu)\}_{\mu=1}^n$ and $\{\overline{\varepsilon}_i\}_{i=1}^m$ for the horizontal and vertical tangent vectors. Here $\overline{\varepsilon}_i$ is the fundamental vector field on P associated to the local vector field $\varepsilon_i \in \mathcal{G}$. The latter comes from using a chart $(\mathcal{O}, \{y_i\}_{i=1}^m)$ on G about the identity e and letting ε_i be the local left-invariant vector field: $\varepsilon_i(a) = \mathrm{d}L_a|_e(\partial/\partial y_i)|_e$. The first set of Cho's formulas [6, p. 2033, formulas (16)–(17)], as well as Kerner's formulas [12, p. 149, formulas (21)–(23)] can be derived from the above by using the bases $\{\partial/\partial\overline{x}_\mu\}_{\mu=1}^n$ and $\{\overline{\varepsilon}_i\}_{i=1}^m$. This is done by using $\sigma(\partial/\partial x_\mu) = (\partial/\partial\overline{x}_\mu)A^i_\mu\overline{\varepsilon}_i$, in (86) and (87) and rearranging. The resulting coordinate expressions for the Ricci tensor in this basis are quite a bit more complicated (and tedious to calculate), and reinforce the value of having global, noncoordinate expressions (86)–(88) for the Ricci tensor (also cf. [4,10]).

Of course the general formulas for the Ricci tensors (52)–(54) and the Kaluza–Klein equations (57)–(59) in Section 6 are valid in the general principal bundle case (h is an arbitrary fiber metric on VP, σ is an arbitrary connection, and the Lie group G is not necessarily semisimple), however, this level of generality should be reduced by requiring that h and σ at least be equivariant so that (1) the quantities in these formulas lose there dependence on the fiber coordinates, and (2) the gauge fields are the customary Yang–Mills fields.

9.2. Warped product bundles

Product bundles, $E = M \times \mathcal{F}$, of semi-Riemannian manifolds (M, g), (F, h_0) form the simplest class of bundles from which one can obtain solutions of the Kaluza–Klein equations (57)–(59). The product structure does not per se give simpler expressions for these equations or for the formulas for the Ricci tensor and scalar. However, simplification does occur if one uses the trivial connection σ.

The *trivial connection* on $E = M \times \mathcal{F}$ is the connection σ defined as follows. For $(x, y) \in M \times \mathcal{F}$ and $X_x \in T_xM$, let $\sigma_{(x,y)}(X_x)$ be the tangent vector in $T_{(x,y)}(M \times \mathcal{F})$ given by

$$\sigma_{(x,y)}(X_x)(\phi) \equiv X_x(\phi_y).$$

Here $\phi : M \times \mathcal{F} \to \mathbb{R}$ is a smooth function and $\phi_y : M \to \mathbb{R}$ is defined by $\phi_y(x') = \phi(x', y)$, for all $x' \in M$. This is easily seen to give a connection and we use the notation:

$$\overline{X} \equiv \sigma(X)$$

for the vector field on $M \times \mathcal{F}$ corresponding to a vector field X on M relative to the trivial connection. $\overline{X}$ is often called the *lift* (or *horizontal lift*) of X to the product bundle (cf. [20, p. 25]). Similarly, each vector field V on $\mathcal{F}$ lifts to a vertical vector field $\overline{V}$ on $M \times \mathcal{F}$. An easy calculation [20] shows that

$$[\overline{X}, \overline{Y}] = [\overline{X, Y}], \qquad [\overline{X}, \overline{V}] = 0, \qquad [\overline{V}, \overline{W}] = [\overline{V, W}]$$

for vector fields X, Y on M and V, W on $\mathcal{F}$. In other notation, the first equation says that $[\sigma(X), \sigma(Y)] = \sigma([X, Y])$ and the second equation says $[\sigma(X), \overline{V}] = 0$ (i.e., $\nabla^\sigma_X\overline{V} = 0$).

Thus, $F(X, Y) = 0$, for all X, Y (identically vanishing gauge fields) and

$$(\nabla^\sigma_X h)(\overline{V}, \overline{W}) = \overline{X}(\overline{V}, \overline{W}) - \nabla^\sigma_X \overline{V} \cdot \overline{W} - \overline{V} \cdot \nabla^\sigma_X \overline{W} = 0.$$

Thus, we easily get the following proposition.

Proposition 4 (warped products with the trivial connection σ). *Suppose $E = M \times \mathcal{F}$ is the product of semi-Riemannian manifolds (M, g), $(\mathcal{F}, h_0)$. Let $h = \rho^* h_0$ be the corresponding fiber metric on the vertical bundle (where $\rho : M \times \mathcal{F} \to \mathcal{F}$ is the projection on $\mathcal{F}$). Then h is gauge-trivial with respect to the trivial connection σ. Note also that $\rho^* = (1 - \sigma\beta)^*$.*

Furthermore, if f is a smooth function on M, then the Ricci tensor and Ricci scalar for the warped product metric:

$$\overline{g} = \pi^* g + (f^2 \circ \pi)(1 - \sigma\beta)^* h_0$$

are given by

$$\overline{\mathrm{Ric}}(\overline{X}, \overline{Z}) = \mathrm{Ric}(X, Z) + \frac{m}{f} H_f(X, Z), \tag{89}$$

$$\overline{\mathrm{Ric}}(\overline{X}, \overline{V}) = 0, \tag{90}$$

$$\overline{\mathrm{Ric}}(\overline{V}, \overline{U}) = \mathrm{Ric}'(V, U) + [f\Delta f + (m-1)|\nabla f|^2](V \cdot W) \tag{91}$$

and

$$\overline{S} = S + \frac{1}{f^2} S' + \frac{2m}{f} \Delta f + \frac{m(m-1)}{f^2} |\nabla f|^2. \tag{92}$$

Here X, Z are vector fields on M and V, W are vector fields on $\mathcal{F}$. Note that all the operations on the right sides of Eqs. (90)–(93) *are with respect to the metrics g on M and h_0 on F.*

The formulas (90)–(93) are the standard ones for a warped product (cf. [20, p. 211]), and are widely used to construct solutions in cosmology of the Einstein equations. More recently, the Randall–Sundrum model [24] has created renewed interest in using warped products in the Kaluza–Klein setting. This model uses the warped product $E = \mathbb{R}^4_f \times \mathbb{R}$, the Minkowski metric on $\mathbb{R}^4$, and Kaluza–Klein metric (in component form):

$$\overline{g} = f^2(y)\eta_{\mu\nu}\,\mathrm{d}x_\mu\,\mathrm{d}x_\nu + r^2\,\mathrm{d}y.$$

Here $r > 0$ is a constant and f has the form $f(y) = \mathrm{e}^{-k|y|}$. Note that to use Eqs. (90)–(93) in this setup, one must consider $E = \mathbb{R}^4_f \times \mathbb{R} \cong \mathbb{R} \times_f \mathbb{R}^4$, i.e., spacetime is the fiber of the bundle. While this is an acceptable, if not awkward, technique here (interchanging fiber and base space), it is inconvenient for including gauge fields in the model and does not apply to the general fiber bundle case (see the following section).

10. Conclusions

In conclusion, we briefly describe a generalization of the Kaluza–Klein metric: $\overline{g} = \pi^* g + (1 - \sigma\beta)^* h$, to one needed for other types of models, such as the Randall–Sundrum model mentioned in Section 9.2. For this it is necessary to include a warp factor α with the spacetime metric g in the higher-dimensional metric: $\overline{g} = \alpha^2\pi^* g + (1 - \sigma\beta)^* h$. Here $\alpha : E \to \mathbb{R}$ is a smooth, positive function. The calculations are complicated considerably by this, even if we restrict to the case where h is gauge-trivial with respect to σ, and so the results and details will be presented in [3]. However the following is one of these results that shows the possibilities for extending the Randall–Sundrum model.

Theorem 7 (field equations for a warped Kaluza–Klein metric). *Suppose P is a principal bundle, h the Killing–Cartan fiber metric on the vertical bundle, and σ a principal connection. For a given smooth, positive function* $\alpha : P \to \mathbb{R}$, *let*

$$\overline{g} = \alpha^2\pi^* g + (1 - \sigma\beta)^* h$$

be the Kaluza–Klein metric. Suppose X, Z are vector fields on M and V, U are vertical vector fields on P. Then the Kaluza–Klein field equations for $\overline{g}$ *are*

$$\begin{aligned}\mathrm{Ric}(X, Z) - \frac{1}{2}S(X \cdot Z) = {} & \frac{1}{8\alpha^2}\langle F, F\rangle(X \cdot Z) - \frac{1}{2\alpha^2}\langle i_X F, i_Z F\rangle \\ & + \left[(n-1)\alpha\Delta'\alpha + \frac{(n-1)(n-2)}{2}|\nabla'\alpha|^2\right](X \cdot Z) \\ & + \left[\frac{(n-2)}{\alpha}\Delta^\sigma\alpha + \frac{(n-2)(n-5)}{2\alpha^2}|\nabla^\sigma\alpha|^2\right](X \cdot Z) \\ & - \frac{(n-2)}{\alpha}H^\sigma_\alpha(X, Z) + \frac{2(n-2)}{\alpha^2}\sigma(X)(\alpha)\sigma(Z)(\alpha) \\ & + \frac{\alpha^2}{2}(S' + \Lambda)(X \cdot Z) + 8\pi T(X, Z), \end{aligned} \tag{93}$$

$$\begin{aligned}(\partial^\sigma F)(X) \cdot V = {} & \frac{(n-4)}{\alpha}\langle \nabla^\sigma\alpha, i_X F\rangle + 2(n-1)\alpha V(\sigma(X)(\alpha)) \\ & -2(n-1)\sigma(X)(\alpha)V(\alpha), \end{aligned} \tag{94}$$

$$\begin{aligned}8\pi T(V, U) = {} & -\frac{1}{4\alpha^4}\langle F \cdot V, F \cdot U\rangle - \frac{1}{8\alpha^4}\langle F, F\rangle(V \cdot U) + \frac{n}{\alpha}H'_\alpha(V, U) \\ & - \left[\frac{n}{\alpha}\Delta'\alpha + \frac{n(n-1)}{2\alpha^2}|\nabla'\alpha|^2\right](V \cdot U) \\ & - \left[\frac{(n-1)}{\alpha^3}\Delta^\sigma\alpha + \frac{(n-1)(n-4)}{2\alpha^4}|\nabla^\sigma\alpha|^2\right](V \cdot U) \\ & - \left[\frac{1}{2\alpha^2}S + \frac{1}{2}\Lambda + \frac{q(m)(m-2)}{8}\right](V \cdot U). \end{aligned} \tag{95}$$

Here the function q is defined by $q(1) = 0$; $q(m) = 1$, *for* $m > 1$.

To see that that the Randall–Sundrum model is a solution of these, note that the Kaluza–Klein metric $\overline{g} = f^2(y)\eta_{\mu\nu}\,\mathrm{d}x_\mu\,\mathrm{d}x_\nu + r^2\,\mathrm{d}y$, is for $\sigma =$ the trival connection, and so $F = 0$

and all the other terms with σ in the notation are zero as well. Also, $S = 0$, $S' = 0$, $q(1) = 0$, and, because the fiber is one-dimensional, $H'_\alpha(V, U) - \Delta'\alpha(V \cdot U) = 0$. Thus the first and the last equations reduce to

$$0 = \left[(n-1)\alpha\Delta'\alpha + \frac{(n-1)(n-2)}{2}|\nabla'\alpha|^2\right](X \cdot Z) + \frac{\alpha^2}{2}\Lambda(X \cdot Z) + 8\pi T(X, Z), \tag{96}$$

$$8\pi T(V, U) = -\frac{n(n-1)}{2\alpha^2}|\nabla'\alpha|^2(V \cdot U) - \tfrac{1}{2}\Lambda(V \cdot U). \tag{97}$$

Since $\alpha(x, y) = f(y) = \mathrm{e}^{-k|y|}$, we have $\nabla'\alpha = r^{-2} f'(\mathrm{d}/\mathrm{d}y) = -kr^{-2} f\mathcal{H}(\mathrm{d}/\mathrm{d}y)$, where $\mathcal{H}$ is the Heaviside function. Thus, $|\nabla'\alpha|^2 = k^2 r^{-2} f^2$, and if we choose k so that $6k^2r^{-2} + \Lambda/2 = 0$, then the right side of (97) is zero. So the stress-energy tensor $T(V, U)$ in the fiber is taken to be identically zero. Further, $\Delta'\alpha = r^{-2} f'' = r^{-2}(k^2 - k\delta_0) f$, where δ_0 is the Dirac delta function at the origin. Using this and the above choice of k, it is easy to see that Eq. (97) reduces to $-3kr^{-2} f^2(X \cdot Z)\delta_0 + 8\pi T(X, Z) = 0$. So the stress-energy tensor $T(X, Z)$ on spacetime is determined from this, and the delta function δ_0 produces the brane for this model.

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